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Homogenization of First Order Differential Operators (**)

SUMMARY. — The problems of homogenization for the system of ordinary differential equations and for the associated transport equations are considered. The properties of Γ limits of some functionals associated to these equations and the connections with the set of rotation vectors of the Poincaré mapping given by the system are investigated.

Omogeneizzazione di operatori differenziali del primo ordine

SUNTO. — Vengono studiati i problemi di omogeneizzazione per sistemi di equazioni differenziali ordinarie e per le equazioni di trasporto associate. Si analizzano le proprietà dei Γ limiti di alcuni funzionali associati a queste equazioni e i collegamenti con l'insieme dei vettori di rotazione delle mappe di Poincaré corrispondenti.

1. - INTRODUCTION

In this paper we study some relationships among the problem of homogenization for the system of ordinary differential equations

$$x' = f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \tag{1}$$

the associated linear transport equations

$$\frac{\partial u}{\partial t} + f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \cdot \frac{\partial u}{\partial x} = 0 \tag{2}$$

and the problem of Γ-convergence of two families of functionals associated to system (1) and equation (2):

$$F_\varepsilon(x) = \int_0^1 \left| x' - f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \right|^2 dt, \qquad G_\varepsilon(u) = \int_\Omega \left| \frac{\partial u}{\partial t} + f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \cdot \frac{\partial u}{\partial x} \right|^2 dt\, dx.$$

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The problem of homogenization of (1) concerns the behaviour of the solutions to the Cauchy problem for (1), as ε tends to zero. It is known that in one space dimension the homogenized equation for (1) always exists and it is of the form $x' = p$ (see [19], [21]). In the space dimension $n > 1$, the homogenized system generally does not exist. The characterization of limits of solutions of (1), as ε tends to zero, is strictly related to the notion of the rotation set of the Poincaré map associated to equation $x' = f(t, x)$. The rotation number was defined by Poincaré in [22] and then extensively studied by many authors (see for example [10], [6], [13], [12] and [18]).

It is possible to show (see [4]) that Γ limits of functionals F_ε and G_ε exist in $L^2(0, 1)$ and $L^2(\Omega)$ topologies, respectively, and are of the form

$$F_0(x) = \int_0^1 \psi(x')\,dt$$

and

$$G_0(u) = \int_\Omega \sum_{i,j=0}^{n} a_{ij}\,\partial_i u\,\partial_j u\,dt\,dx\,,$$

where ψ is a nonnegative, convex function on R^n and (a_{ij}) is a $(n+1) \times (n+1)$ constant, symmetric and positive semidefinite matrix. The family of functionals G_ε was considered also by De Giorgi in [8].

We prove, in the one dimensional case, that the function ψ has exactly one zero and it is p, the rotation number of (1). We also show that functional G_0 is of the form $k \int_\Omega |u_t + p u_x|^2\,dt\,dx$, and the constant k is positive iff the Poincaré map associated to $x' = f(t, x)$ has an absolutely continuous (with respect to Lebesgue measure) invariant measure with L^2 density. If f in (1) depends only on x (autonomous case), it is possible to calculate the value of k and $k > 0$ if $f \equiv 0$ or $f(x) \neq 0$ for every $x \in R$. The vectorial case ($n > 1$) is much more complicated. In this situation, it is proved that the set of zero points of ψ contains the set of rotation vectors.

In the last section we will show that for the two dimensional system

$$\begin{cases} x' = 0\,, \\ y' = g\left(\dfrac{x}{\varepsilon}\right), \qquad x, y \in R,\ g \in C^1(R) \text{ and periodic}\,, \end{cases}$$

the homogenized equation does not exist. Let us underline that in this case the sequence of solutions of the corresponding first order linear hyperbolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + g\left(\dfrac{x}{\varepsilon}\right) \cdot \dfrac{\partial u}{\partial y} = 0\,, \\ u(0, x, y) = \phi(x, y)\,, \qquad \phi \text{ given} \end{cases}$$

is weakly convergent. The identification of the limit equation, as ε tends to zero, is due to Tartar (see [23], [24]). The case when the function g depends also on t was considered in [2]. Some generalizations to the transport equations in R^n can be found in [3]. Another generalization was recently given by Hou and Xin in [15]. Precisely, they proved that in the two-dimensional case, if $\operatorname{div} f = 0$, $f \neq 0$, then the solutions of the first order linear equations (2) satisfying the initial condition $u(0, x) = \phi(x)$ always weakly converge to a function which is not, in general, a solution of an equation of the same type. Finally, we remark that the linear transport equations have been studied by DiPerna and Lions in [11] and related homogenization problems have been considered by Mascarenhas in [16], [17].

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2. - Preliminaries

In this section we introduce notations and definitions which will be used throughout the paper.

Let R^n ($n \geq 1$) be the n-dimensional Euclidean space and let m be the Lebesgue measure in R^n. The norm and the inner product of R^n is denoted by $|\,|$ and «·», respectively. We write $I = [0, 1]$.

By a I^n-periodic function we mean a function on R^n which is periodic with period I^n, i.e. with period 1 in each variable. If E is an open subset of R^n, for $r = 0, 1, \ldots$ and $k = 1, 2, \ldots$ we denote by $C^r(E)^k$ the space of functions from E into R^k that are continuously differentiable up to the r-th order, $C^r(E)^1 = C^r(E)$. The space $C^r(E)^k$ will be endowed with the topology of $C^0(E)^k$ i.e. the topology of uniform convergence of functions on compact sets. By $C^r_{\mathrm{per}}(R^n)^k$ we indicate the subspace of $C^r(R^n)^k$ of all I^n-periodic functions. $C^1_0(R^n)$ denotes the space of functions in $C^1(R^n)$ having compact supports.

If $f\colon R^n \to R^k$ is a differentiable function, then the notation $Df(x)$ stands for the $k \times n$ Jacobian matrix of f at x. For $f\colon R^r \to R$, we set $\{f = 0\} = \{\lambda \in R^r \,|\, f(\lambda) = 0\}$.

For any subset A of R^n we denote by $\operatorname{conv}(A)$ the convex hull of A. If X is a Banach space, we denote the strong and the weak topology in X, by $s - X$, $w - X$, respectively. The vectors (e_i) represent the canonical basis of R^n. For each $x \in R^n$, we write $x^{\sim} = x - [x]$, where $[x]$, is the vector composed of the integer parts of the components of x.

Given $f \in C^1_{\mathrm{per}}(R^{n+1})^n$ and $\varepsilon > 0$, we denote by $T_t^{(\varepsilon)}(x_0)$ the value of the solution to the problem

$$x' = f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right), \tag{3}$$

$$x(0) = x_0 \tag{4}$$

at the time t. It is well known (see for example [6], [14]) that $T_t^{(\varepsilon)}$ is a diffeomorphism from R^n to R^n and the following properties hold:

$$(5)\quad \begin{cases} T_t^{(\varepsilon)}(x) = \varepsilon T_{t/\varepsilon}\left(\frac{x}{\varepsilon}\right), & \forall t \in R,\ \forall x \in R^n, \\ T_t(x+z) = T_t(x) + z, & \forall t \in R,\ \forall x \in R^n,\ \forall z \in Z^n, \\ T_{t+k} = T_t \circ T_k, & \forall t \in R,\ k \in Z, \end{cases}$$

where $T_t = T_t^{(1)}$.

We also write T^n, $T^n = T \circ T^{n-1}$ for each $n \in N$, where T^0 denotes the identity mapping, $T = T_1^{(1)}$ and T^{-n} stands for the inverse of T^n. The mapping $T_1^{(\varepsilon)}$ is called Poincaré map associated to (3).

We admit the following

Definition 2.1 (see [19], [20], [21]): *We say, that the family of systems* (3), (4), *G-converges to the system*

$$(6)\qquad x' = p,$$

if for every $x_0 \in R^n$ the solutions $T_t^{(\varepsilon)}(x_0)$ converge, as $\varepsilon \to 0$, uniformly with respect to t on compact intervals to the solution of initial problem (6), (4). *Moreover, if this convergence is also uniform with respect to the initial value x_0 from the compact subset of R^n, then we say that* (3) *strongly G-converges to* (6). *The vector p will be called the homogenized value of the function f.*

Remark 2.1: It is proved (see [22], [10], [6], [21]) that in dimension one the G-limit always exists and for every $x_0 \in R^n$ we have

$$p = \lim_{t \to \infty} \frac{T_t(x_0)}{t}.$$

It is easy to observe that if $n = 1$, then the G-convergence is equivalent to the strong G-convergence.

It is possible to give explicit formulae on the number p, only in some particular cases. For example, if $f(t,x) = g(t)h(x)$, then $p = M(g)N(h)$, where $M(g) = \int_I g(s)\,ds$ and either $N(h) = (M(1/h))^{-1}$ if $h(x) \neq 0$ for every $x \in I$ or $N(h) = 0$ if there exists $x_0 \in I$ such that $h(x_0) = 0$.

In dimension $n > 1$, it is not true, in general, that (3) G-converges (see Example 4.2 in Section 4).

Definition 2.2 (see [14], [18]): *A vector p of R^n is called a rotation vector of the Poincaré map T if there exist a sequence $\{p_k\} \subset R^n$ and a subsequence n_k of integers such that*

$$p = \lim_{k \to \infty} \frac{T^{n_k}(p_k) - p_k}{n_k}.$$

The set of all rotation vectors will be denoted by $\rho(T)$.

From the results of [18] and [12] we can easily get the following properties of $\rho(T)$.

PROPOSITION 2.1: *a) The set $\rho(T)$ is a non empty, connected and compact subset of $\boldsymbol{R}^n$. It is contained in the convex hull of the set*

$$\rho_1(T) = \left\{ p \in \boldsymbol{R}^n :\ \exists p_0 \in \boldsymbol{R}^n,\ p = \lim_{n \to \infty} \frac{T^n(p_0)}{n} \right\}. \tag{7}$$

b) If $n = 1$, then the set $\rho(T)$ consists of only one real number (i.e. $\rho(T) = \{p\}$ and in this case we also write $\rho(T) = p$).

c) If $n = 2$ and T comes from a function f which does not depend on t, then the set $\rho(T)$ is contained in a line through the origin.

In the one dimensional case, we say that $\rho(T)$ belongs to the class $\mathcal{K}$, if the rotation number of the Poincaré map T is quadratic irrational (i.e. it is of the form $\alpha \pm \sqrt{\beta}$ with $\alpha, \beta \in \boldsymbol{Q}$).

Let us recall now the following result which will be the most important tool in the demonstration of Lemma 3.1.

PROPOSITION 2.2 (see [14]): *If $f \in C^4_{per}(\boldsymbol{R}^2)$ and $\rho(T) \in \mathcal{K}$, then T is conjugate to a translation by a diffeomorphism g: $\boldsymbol{R} \to \boldsymbol{R}$ with $g, g^{-1} \in C^1$, such that $g(x+1) = g(x) + 1$ $\forall x \in \boldsymbol{R}$, i.e. $T = g^{-1} \circ R_\gamma \circ g$, where $R_\gamma(x) = x + \gamma$ and $\gamma = \rho(T)$.*

REMARK 2.2: If f satisfies the hypotheses of Proposition 2.2, then maps $\{T^n\}_{n \in Z}$ are equilipschitzian and this is equivalent to the fact that there exist two positive constants C_1, C_2 such that $C_1 \leq D_x T_t(x) \leq C_2$ for each $t \in \boldsymbol{R}$, $x \in \boldsymbol{R}$.

REMARK 2.3 (cf. [6], [13], [14]): Let $n = 1$. Then the function $f \mapsto \rho(T)$ is continuous from C^1_{per} (equipped with the C^0-topology) into $\boldsymbol{R}$ and it is increasing (i.e. if $f \leq g$, then $\rho(T_f) \leq \rho(T_g)$).

Now we pass to functionals for which we consider the following notion of Γ-convergence:

DEFINITION 2.3 (see [7]): *Let (X, τ) be a metric space and let $(F_\varepsilon)_{\varepsilon > 0}$ be a family of real functions defined on X into $\overline{\boldsymbol{R}}$. We say that (F_ε), $\Gamma(\tau - X)$-converges to F_0 and we write*

$$F_0 = \Gamma(\tau - X^-) \lim_{\varepsilon \to 0} F_\varepsilon$$

iff for every $x \in X$, for every sequence ε_h which tends to 0, it holds

a) for every sequence (x_h) converging to x

$$F_0(x) \leq \liminf_{h \to \infty} F_{\varepsilon_h}(x_h),$$

b) there exists a sequence (x_k) converging to x such that

$$F_0(x) = \lim_{k \to \infty} F_{\varepsilon_k}(x_k).$$

Let us consider now two families of functionals associated to the system (1) and equation (2):

$$F_\varepsilon(x) = \int_I \left| x' - f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \right|^2 dt, \quad x \in C^1(I)^n, \tag{8}$$

$$G_\varepsilon(u) = \int_\Omega \left| u_t + f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \cdot u_x \right|^2 dt\, dx, \quad x \in C^1(\Omega), \tag{9}$$

where Ω is a bounded open subset of $\boldsymbol{R}^{n+1}$. The functionals G_ε are quadratic but they are not positive definite.

From the result of [4], we obtain two propositions which we will need in the next sections.

Proposition 2.3: *If $f \in C^1_{per}(\boldsymbol{R}^{n+1})^n$, F_ε are given by (8), then there exists*

$$F_0 = \Gamma(s - L^2(I)^-) \lim_{\varepsilon \to 0} F_\varepsilon, \tag{10}$$

and

$$F_0(x) = \int_I \psi(x')\, dt, \quad x \in C^1(I)^n, \tag{11}$$

where ψ is a nonnegative, convex function defined on $\boldsymbol{R}^n$.

Moreover, the function ψ is given by the formula

$$\psi(\xi) = \lim_{T \to +\infty} \psi_T(\xi) \tag{12}$$

where

$$\psi_T(\xi) = \inf\left\{ \frac{1}{T} \int_0^T |x' - f(t,x)|^2 dt \colon x(0) = 0,\ x(T) = \xi T,\ x \in C^1([0,T])^n \right\}. \tag{13}$$

The function ψ is not in general quadratic, as it was shown in [5].

Proposition 2.4: *If $f \in C^1_{per}(\boldsymbol{R}^{n+1})$, G_ε are given by (9), then there exists*

$$G_0 = \Gamma(s - L^2(\Omega)^-) \lim_{\varepsilon \to 0} G_\varepsilon, \tag{14}$$

and functional G_0 is of the form

$$G_0(u) = \int_\Omega A\, Du \cdot Du\, dt\, dx, \quad u \in C^1(\Omega). \tag{15}$$

The matrix A is constant, symmetric, positive semidefinite and for every $\xi \in \mathbf{R}^{n+1}$

$$A\xi \cdot \xi = \inf\{J(u)\colon\ u = \xi \cdot z + \phi(z),\ z = (t, x),\ \phi \in C^1_{\mathrm{per}}(\mathbf{R}^{n+1})\}, \tag{16}$$

where

$$J(u) = \int_{Y^{n+1}} |u_t + f(t, x) \cdot u_x|^2 \, dt\, dx. \tag{17}$$

3. - The case of dimension 1

In this section we assume that $n = 1$ and we study the connections among the homogenization problems for (3) and (2), and the Γ-convergence of the functionals F_ε, G_ε.

Let $f \in C^1_{\mathrm{per}}(\mathbf{R}^2)$ and x^ε be the solutions to the problem (3), (4). We known (see Remark 2.1 and Proposition 2.1) that for each $x_0 \in \mathbf{R}$

$$x^\varepsilon \to x^0 \quad \text{in } C(I), \tag{18}$$

where $x^0(t) = pt + x_0$ and $\rho(T) = \{p\}$.

Lemma 3.1: *Let $f \in C^1_{\mathrm{per}}(\mathbf{R}^2)$. If $b_\varepsilon \to 0$ in $L^1(I)$ weakly, then the sequence of equations*

$$y' = f\left(\frac{t}{\varepsilon}, \frac{y}{\varepsilon}\right) + b_\varepsilon(t) \tag{19}$$

strongly G-converges to equation (6).

Proof: In order to prove the lemma, let y^ε, $\varepsilon > 0$ be the solutions of (19) such that $y^\varepsilon(0) = r_\varepsilon$, where $r_\varepsilon \to x_0$. We will show that $y^\varepsilon \to x^0$ in $C(I)$. Let v^ε, $\varepsilon > 0$ be the solutions to the problem

$$\begin{cases} v_t(t,x) + f\left(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}\right) v_x(t,x) = 0, & t \in \mathbf{R},\ x \in \mathbf{R}, \\ v(0,x) = x, & x \in \mathbf{R}. \end{cases}$$

The solutions x^ε of (3), (4) satisfy the relation $v^\varepsilon(t, x^\varepsilon(t)) = x_0$ for each $t \in \mathbf{R}$. Now, we have the following relations

$$a := v^\varepsilon(t, y^\varepsilon(t)) - v^\varepsilon(t, x^\varepsilon(t)) = (y^\varepsilon(t) - x^\varepsilon(t))\, v^\varepsilon_y(t, \theta), \tag{20}$$

where $\theta \in \mathbf{R}$ and

$$\begin{aligned} a &= \int_0^t \left(v^\varepsilon_t(s, y^\varepsilon(s)) + v^\varepsilon_y(s, y^\varepsilon(s)) \left(f\left(\frac{s}{\varepsilon}, \frac{y^\varepsilon(s)}{\varepsilon}\right) + b_\varepsilon(s) \right) \right) ds + r_\varepsilon - x_0 = \\ &= \int_0^t v^\varepsilon_y(s, y^\varepsilon(s))\, b_\varepsilon(s)\, ds + r_\varepsilon - x_0. \end{aligned} \tag{21}$$

Since $v^\varepsilon(t, z) = (T_t^{(\varepsilon)})^{-1}(z)$, for $z, t \in \boldsymbol{R}$, it holds

$$v_z^\varepsilon(t, z) = D_u((T_{t/\varepsilon})^{-1}(u))|_{u = z/\varepsilon}. \tag{22}$$

Now, we suppose additionally that $f \in C_{per}^4(\boldsymbol{R}^2)$ and $\rho(T) \in \mathcal{K}$. We know (see Remark 2.2) that under this assumption there exists a constant $c > 0$ such that $c^{-1} \leq D_z(T^n(z)) \leq c$ for each $n \in \boldsymbol{Z}$ and $z \in \boldsymbol{R}$. These estimations, relations (20), (21), (22) and the fact that $r_\varepsilon \to x_0$ imply

$$\|y^\varepsilon - x^\varepsilon\|_{C(I)} \to 0 .$$

Hence and from (18), we get $y^\varepsilon \to x_0$ in $C(I)$.

Now, let us return to the general case: $f \in C_{per}^1(\boldsymbol{R}^2)$. Let $\hat{\varepsilon} > 0$. By Remark 2.3, there exist $f_1, f_2 \in C_{per}^4(\boldsymbol{R}^2)$ such that $f_1 \leq f \leq f_2$ and

$$p - \hat{\varepsilon} \leq p_1 \leq p \leq p_2 \leq p + \hat{\varepsilon}, \tag{23}$$

where $p_i = \rho(T_{f_i})$, $p_i \in \mathcal{K}$, $i = 1, 2$.

Let z_i^ε, $i = 1, 2$, $\varepsilon > 0$ be the solutions of the problems

$$z' = f_i\left(\frac{t}{\varepsilon}, \frac{z}{\varepsilon}\right) + h_\varepsilon(t), \qquad z(0) = r_\varepsilon, \qquad i = 1, 2 .$$

By the properties of f_1 and f_2 we have $z_1^\varepsilon(t) \leq y^\varepsilon(t) \leq z_2^\varepsilon(t)$, $t \in I$ and by the first part of the lemma, we know that $z_i^\varepsilon(t) \to p_i t + x_0$, $i = 1, 2$ uniformly on I. Because $\hat{\varepsilon}$ was arbitrary, hence and from (23) it follows that $y^\varepsilon(t) \to pt + x_0$ uniformly on I. ■

EXAMPLE 3.1: It is easy to see, using Lemma 3.1 that the sequence $x' = f(t/\varepsilon) + g(x/\varepsilon)$ strongly G-converges to the equation $x' = N(M(f) + g)$.

Given a function $f \in C_{per}^1(\boldsymbol{R}^2)$, we consider now the functionals F_ε, F_0, given by (8), (11). We know (see Proposition 2.3) that the sequence of functionals F_ε Γ-converges to the functional F_0 of the form (11) with a convex, nonnegative function ψ given by (12).

THEOREM 3.1: *Let $n = 1$. Then $\rho(T) = \{\psi = 0\}$.*

PROOF: Let, as before, x^ε, x^0 be the solutions to (3), (4) and to (6), (4), respectively. We know that $x^\varepsilon \to x^0$ uniformly on compact sets and $\rho(T) = \{p\}$. Since $F_\varepsilon(x^\varepsilon) = 0$ we have $F_0(x^0) = 0$ and by (10) we get $\psi(p) = 0$.

Now, we are going to prove that p is a unique zero point of ψ. Let us suppose $\psi(\lambda) = 0$, $\lambda \in \boldsymbol{R}$. Then $F_0(y^0) = 0$, where $y^0(t) = \lambda t + x_0$. From (10), we can find a sequence $y^\varepsilon \in C^1(I)$, $\varepsilon > 0$ such that

$$y^\varepsilon \to y^0 \quad \text{in } L^2(I) \tag{24}$$

and $\lim_{\varepsilon \to 0} F_\varepsilon(y^\varepsilon) = 0$. Let us denote

$$g_\varepsilon(t) = y^{\varepsilon\prime}(t) - f\left(\frac{t}{\varepsilon}, \frac{y^\varepsilon(t)}{\varepsilon}\right), \qquad t \in I .$$

Since $g_\varepsilon \to 0$ in $L^2(I)$ and f is bounded, the sequence $(y^{\varepsilon'})$ is bounded in $L^2(I)$. This property and (24) imply $y^\varepsilon \to y^0$ in $C(I)$. So, in particular $y^\varepsilon(0) \to x_0$. Now, by virtue of Lemma 3.1, we have that $y^\varepsilon(t) \to pt + x_0$ for each t. Hence $\lambda = p$. ∎

Next, let us consider the family of functionals G_ε given by (9).
We have the following

THEOREM 3.2: *Let $f \in C^1_{per}(R^2)$. Then, there exists a constant $k \in [0, 1]$ such that the sequence (G_ε) $\Gamma(s - L^2(\Omega))$-converges to the functional*

$$G_0(u) = k \int_\Omega |u_t(t,x) + p \cdot u_x(t,x)|^2 \, dt\, dx\,, \tag{25}$$

where p is the rotation number of T.

Since in the case $n = 1$, the G-convergence is equivalent to the strong G-convergence, the Theorem 3.2 is a particular case of Theorem 4.1.

From formulae (16), (17), and (25), it follows that

$$k = Ae_1 \cdot e_1\,. \tag{26}$$

In the autonomous case the following proposition gives an explicit formula for the constant k.

PROPOSITION 3.1: *Let $f \in C^1_{per}(R)$ (f does not depend on t) and $f(x) \neq 0$ for each $x \in R$. Then $k = N(f^2)(N(f))^{-2}$.*

PROOF: Let us consider the case $f > 0$. To calculate k, we use (26). Since we deal with a convex functional, every solution to the Euler equation is also a minimum point. In our case the Euler equation is the following

$$(u_t + f(x)u_x)_t + (f(x)u_t + f^2(x)u_x)_x = 0\,.$$

We look for its solutions in the form $u(t,x) = t + \psi(x)$ with $\psi \in C^1_{per}(R)$. We have $f(x) + f^2(x)\psi'(x) = c$. The periodicity of ψ implies $\int_I \psi'(t) = 0$ and then $c = N(f^2)N(f)^{-1}$. Calculating the value of J given by (17) at the solution of Euler equation, we find

$$k = \int_I (1 + f(x)\psi'(x))^2 dx = N(f^2)N(f)^{-2}. \quad ∎$$

REMARK 3.1: If $f \in C^1_{per}(R)$ (f does not depend on the space variable x), then $k = 1$. In particular, $f \equiv 0$ implies $k = 1$.

REMARK 3.2: If $f \in C^1_{\text{per}}(R)$ (f does not depend on t) and $f(x) = 0$ on E, $m(E) > 0$, then $k > 0$. In fact from (16), (17) we have

$$k \geqslant \inf\left\{ \int_{I \times E} |u_t(t,x)|^2 \, dt \, dx \,\middle|\, u(t,x) = t + \phi(t,x), \ \ \phi \in C^1_{\text{per}}(R^2) \right\} \geqslant m(E).$$

PROPOSITION 3.2: *If $f \in C^1_{\text{per}}(R)$ (f does not depend on t), $f(0) = 0$, $f(x) > 0$ in I and $f(x) \sim x^\alpha$ when $x \to 0$ with $\alpha \geqslant 1$, then $k = 0$.*

PROOF: We observe that

$$k \leqslant \inf\left\{ \int_I (1 + f(x)\psi'(x))^2 \, dx \,\middle|\, \psi \in C^1_{\text{per}}(R) \right\}$$

and we will show that the above infimum is equal to zero. Indeed, it is enough to take $\forall \delta \in (0, 1/2)$, $\psi_\delta \in C^1(I)$ such that ψ_δ is symmetric with respect to $x = 1/2$, $\psi_\delta(0) = \psi_\delta(1) = 0$ and $\psi'_\delta = 1/f$ in $(\delta, 1 - \delta)$ (with some positive small δ). Over the intervals $(0, \delta)$ and $(1 - \delta, 1)$ we define ψ_δ in such a way that

$$\lim_{\delta \to 0} \left(\int_0^\delta (1 + f(x)\psi'_\delta(x))^2 \, dx + \int_{1-\delta}^1 (1 + f(x)\psi'_\delta(x))^2 \, dx \right) = 0.$$

The last condition is easily satisfied because of the assumptions on the function f. Therefore the above infimum is equal to zero and hence $k = 0$. ■

Now, our next objective is to obtain a sufficient and necessary condition which will ensure that $k > 0$. We have

THEOREM 3.3: *Let $f \in C^1_{\text{per}}(R^2)$. Then, $k > 0$ iff T possesses an absolutely continuous with respect to m invariant measure with density in $L^2_{\text{loc}}(R)$.*

In order to prove Theorem 3.3, we put

$$J_0(h) = \int_I (1 + h(Ty) - h(y))^2 \, dy, \qquad h \in C^1_{\text{per}}(R),$$

$$c_1 = \min\{D_y T_t(y) \mid (t, y) \in I^2\}, \qquad c_2 = \max\{D_y T_t(y) \mid (t, y) \in I^2\}$$

and we need the following

LEMMA 3.2: *If $f \in C^1_{\text{per}}(R^2)$, $k_1 = \inf\{J_0(h) \mid h \in C^1_{\text{per}}(R)\}$, then*

$$c_1 k_1 \leqslant k \leqslant c_2 k_1.$$

PROOF: Recalling that the function $y \mapsto T_t(y)$ is increasing and $D_y T_t \in C^1(R^2)$, we have $c_1 > 0$, $c_2 > 0$. Firstly, we will show that $c_1 k_1 \leqslant k$. By formulae (16), (17) and (26), we have

$$J(u) = \int_{\tilde{I}^2} |u_t(t, T_t y) + f(t, T_t y) u_x(t, T_t y)|^2 D_y T_t(y)\, dt\, dy ,$$

where $\tilde{I}^2 = \{(t, y)\colon\ t \in I,\ \ T_t(y) \in I\}$.

Putting $w(t, y) = u(t, T_t(y))$ and using the Cauchy-Schwarz inequality, we find that

$$J(u) = \int_{\tilde{I}^2} w_t^2(t, y) D_y T_t(y)\, dt\, dy \geq c_1 \int_{\tilde{I}^2} w_t^2(t, y)\, dt\, dy =$$

$$= c_1 \int_{I^2} w_t^2(t, y)\, dt\, dy \geq c_1 \int_I (w(1, y) - w(0, y))^2\, dy ,$$

because $w(t, \cdot)$ is I-periodic.

Now, since $w(t, y) = t + \phi(t, T_t(y))$ with $\phi \in C^1_{per}(R^2)$, putting $h(y) = \phi(0, y)$, we obtain $c_1 k_1 \leq k$.

To prove the second inequality, we note that

$$k = \inf\{J(u) \,|\, u \in \mathfrak{M}\}, \qquad \mathfrak{M} = \{u \,|\, u(t, x) = t + \phi(t, x),\ \ \phi \in \mathrm{Lip}_{per}(R^2)\},$$

where $\mathrm{Lip}_{per}(R^2)$ denotes the periodic Lipschitz functions defined on R^2. Next, for each $h \in C^1_{per}(R)$ we construct $u(h) \in \mathfrak{M}$ such that

$$J(u(h)) \leq c_2 J_0(h) . \tag{27}$$

Namely, we put $w(t, y) = h(y)(1 - t) + t(h(Ty) + 1)$ for $t \in I$, $y \in R$ and next we extend w on the whole R^2 by formula

$$w(t, y) = [t] + w(t - [t], T^{[t]} y) .$$

Finally, we define $u(h)(t, y) = w(t, T_t^{-1} y)$. The inequality (27) implies $k \leq c_2 k_1$.

Remark 3.3: From the proof of Lemma 3.2 it is easy to observe that

$$k_1 = \inf\{J_0(h) \,|\, h \in L^2_{loc}(R),\ \ h \text{ is } I\text{-periodic}\} .$$

Proof of Theorem 3.3: By Lemma 3.2 $k = 0$ iff $k_1 = 0$. The last fact is equivalent to $1 \in \overline{\mathrm{Im}\, U}$, where $U\colon L^2(I) \to L^2(I)$, $Uh = h \circ T - h$. Now, since $\overline{\mathrm{Im}\, U} = (\ker U^*)^\perp$, we have

$$1 \in (\ker U^*)^\perp \Leftrightarrow \ker U^* \subset \left\{h \in L^2(R)\colon \int_I h(y)\, dy = 0\right\}.$$

Hence, $k > 0$ iff there exists $h \in \ker U^*$ such that $\int_I h(y)\, dy \neq 0$, so iff there exists $h \in \ker U^*$, $h \geq 0$ a.e. and $\int_I h(y)\, dy = 1$. Now, we observe that $h \in \ker U^*$ iff $h \in L^2(I)$

and $b(T^{-1})DT^{-1} = b$. The last equality means that $b(y)dy$ is T invariant measure. ■

4. - THE CASE OF DIMENSION n

In this part we turn our attention to the vectorial case and we present some generalizations of the results obtained in the previous section.

We start with the following

PROPOSITION 4.1: *Let $f \in C^1_{per}(R^{n+1})^n$. Suppose that the sequence of equations* (3) *G-converges to the equation* (6). *Then $\psi(p) = 0$.*

PROOF: From the G-convergence it follows that the sequence (x^ε) of the solutions to (3), (4) converges uniformly on I to the solution x^0 of (6), (4). Therefore from the definition of Γ-limit $F_0(x^0) = 0$, which gives $\psi(p) = 0$, because ψ is nonnegative. ■

REMARK 4.1: If $f \in C^1_{per}(R)^n$ (f does not depend on the space variable x), then we can explicitly calculate the function ψ given by formulae (12), (13) and we get $\psi(\xi) = |\xi - M(f)|^2$, where $M(f) = (M(f_1), \ldots, M(f_n))$.

LEMMA 4.1: *Let $f \in C^1_{per}(R^{n+1})^n$ and for each $t \in R^+$, let r_t be a point in I^n. If for each sequence $(t_k)_{k \in N}$, the sequence $((1/k)\, T^k r_{t_k})_{k \in N}$ converges to p, then $\lim_{\tau \to \infty} T_\tau(r_\tau)/\tau = p$.*

The proof of the lemma, being simple, is omitted.

In the n-dimensional case we have the following relationship between the G-convergence and the rotation set for T.

PROPOSITION 4.2: *Let $f \in C^1_{per}(R^{n+1})^n$. The sequence of equations* (3) *strongly G-converges to* (6) *iff $\rho(T) = \{p\}$.*

PROOF: Suppose that $\rho(T) = \{p\}$. Let $r^\varepsilon \to x_0$. Immediately from the definition of $\rho(T)$ and our assumption, it follows that for each bounded sequence (p_n) we have $T^n(p_n)/n \to p$. Hence and by Lemma 4.1, we get

$$T_t^{(\varepsilon)}(r^\varepsilon) = t\,\frac{T_{t/\varepsilon}((r^\varepsilon/\varepsilon)^-)}{t/\varepsilon} + \varepsilon\left[\frac{r^\varepsilon}{\varepsilon}\right] \to x_0 + pt,$$

for each $t \in I$. Since the set of solutions is compact in $C(I)^n$ the last convergence is uniform on I.

To prove the converse, we assume that the equation (3) strongly G-converges to (6) and $q \in \rho(T)$. Then there exist sequences $(x_i) \subset R^n$, $n_i \to \infty$ such that

$$\frac{T^{n_i}(x_i) - x_i}{n_i} \to q.$$

By the strong G-convergence, we have

$$\text{(28)} \qquad \frac{T^{n_i}(x_i) - x_i}{n_i} = T_1^{(1/n_i)}\left(\left(\frac{x_i}{n_i}\right)^{-}\right) + \left[\frac{x_i}{n_i}\right] - \frac{x_i}{n_i} \to p\,,$$

for every convergent subsequence of $(x_i/n_i)_i^{-}$. Then all the sequence (28) converges and hence $p = q$. ■

The following proposition establishes a relationship between the set $\rho(T)$ and the Γ-limit functional (11).

Proposition 4.3: *Let $f \in C^1_{per}(R^{n+1})^n$, F_ε, F_0, ψ be given by* (8), (11), (12). *Then* $\operatorname{conv}(\rho(T)) \subset \{\psi = 0\}$.

Proof: Since the set $\{\psi = 0\}$ is convex, by Proposition 2.1*a*), it is enough to prove that if $p \in \rho_1(T)$ (see (7)) then $p \in \{\psi = 0\}$.

Let p_0 be a vector such that $p = \lim\limits_{n \to \infty} \dfrac{T^n(p_0)}{n}$. We are going to construct a sequence of solutions $(T_t^{(\varepsilon)}(\varepsilon p_0))_{\varepsilon > 0}$ of (3) which converges to a solution of (6).

Firstly, let us observe that for $s = m/k$, we can find a subsequence of $(T_s^{(\varepsilon)}(\varepsilon p_0))$, converging to $x^0(s) = ps$. In fact, putting $\varepsilon_l' = 1/(kl)$, by assumptions on p_0, p, we have

$$\text{(29)} \qquad T_s^{(\varepsilon_l')}(\varepsilon_l' p_0) = \frac{m}{k}\,\frac{T^{ml}(p_0)}{ml} \to ps.$$

Now, using the diagonal argument, we can construct a subsequence $(\varepsilon_l'') \subset (\varepsilon_l')$ such that the sequence (29) converges for each $s \in Q$. Since the set of solutions $(T_t^{(\varepsilon_l'')}(\varepsilon_l'' p_0))$ is compact in $C(I)^n$, then passing to the next subsequence (ε_l''') we have $x^{\varepsilon_l'''}$ converges uniformly on I to x^0. Since $F_{\varepsilon_l'''}(x^{\varepsilon_l'''}) = 0$, then $F_0(x^0) = 0$ and this implies $\psi(p) = 0$. ■

Given a function $\phi \in C_0^1(\boldsymbol{R}^n)$, we consider now the system of first order partial differential equations corresponding to (3)

$$\text{(30)} \qquad \begin{cases} u_t + f\left(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}\right)\cdot u_x = 0\,, & t \in \boldsymbol{R},\ x \in \boldsymbol{R}^n, \\ u(0, x) = \phi(x)\,, & x \in \boldsymbol{R}^n. \end{cases}$$

We have

Lemma 4.2: *Let $f \in C^1_{per}(R^{n+1})^n$. Suppose that the sequence of equations* (3) *strongly G-converges to the equation* (6). *Then the sequence (u_ε) of solutions to* (30) *converges in $L^2(\Omega)$ to a function u_0 which is solution to*

$$\begin{cases} u_t + p\cdot u_x = 0\,, & t \in \boldsymbol{R},\ x \in \boldsymbol{R}^n, \\ u(0, x) = \phi(x)\,, & x \in \boldsymbol{R}^n. \end{cases}$$

where p is given by (6).

PROOF: By the definition of the strong G-convergence of (3) to (6), we have $T_t^{(\varepsilon)}(x) \to x + pt$, uniformly on compact subsets of $\mathbf{R}^{n+1}$. Hence, $(T_t^{(\varepsilon)})^{-1} \to x - pt$, in the same sense, and finally $u^\varepsilon(t,x) = \xi((T_t^{(\varepsilon)})^{-1}(x))$ tends in $L^2_{\mathrm{loc}}(\mathbf{R}^{n+1})$ to $u^0(t,x) = \xi(x - pt)$. ■

REMARK 4.2: In the case when f does not depend on t, Lemma 4.2 remains true even if the hypothesis of the strong G-convergence is replaced by the G-convergence.

EXAMPLE 4.1: It can be shown that if $f \in C^2_{\mathrm{per}}(\mathbf{R}^n)^n$, then the equations $x' = \nabla f(x/\varepsilon)$ strongly G-converges to the equation $x' = 0$, ∇f denotes the gradient of f.

Our next aim is to show the relationship between G-convergence of equations (3) and Γ-convergence of functionals (9).

THEOREM 4.1: *Let $f \in C^1_{\mathrm{per}}(\mathbf{R}^{n+1})^n$. If the sequence of equations* (3) *strongly G-converges to the equation* (6), *then there exists a constant $k \in [0, 1]$ such that the sequence of functionals* (G_ε) *given by* (9) *is $\Gamma(s - L^2(\Omega)^-)$-convergent to the functional*

$$G_0(u) = k \int_\Omega |u_t(t,x) + p \cdot u_x(t,x)|^2 \, dt \, dx,$$

where p is given by the equation (6).

PROOF: We known (see Proposition 2.4) that the $\Gamma(s - L^2(\Omega)^-)$-limit of (G_ε) exists and it is a nonnegative quadratic functional of the form (15), where A is $(n + 1) \times (n + 1)$ constant, positive semidefinite and symmetric matrix (we denote $Du = (u_t, u_{x_1}, \ldots, u_{x_n}) = (u_t, u_x)$).

By Lemma 4.2, the sequence of functions $u^\varepsilon(t,x) = \xi((T_t^{(\varepsilon)})^{-1}(x))$ converges in $L^2(\Omega)$ to $u^0(t,x) = \xi(x - pt)$ for each $\xi \in C^1_0(\mathbf{R}^n)$. Since $G_\varepsilon(u^\varepsilon) = 0$, by Γ-convergence $G_0(u^0) = 0$. The last equality means that $A\,Du \cdot Du = 0$ a.e. for each regular solution u of the equation $u_t + p \cdot u_x = 0$. From the arbitrarity of ξ, we deduce that $Aw \cdot w = 0$ for each $w \in \mathbf{R}^{n+1}$, $w = (-a \cdot p, a)$, $a \in \mathbf{R}^n$. This means that $Ax \cdot x = 0$ for each $x = (x_0, x_1) \in \mathbf{R}^{n+1}$, $x_1 \in \mathbf{R}^n$ such that $x_0 + p \cdot x_1 = 0$.

Now, we define the $(n + 1) \times (n + 1)$ matrix $P = \begin{pmatrix} 1 \\ p \end{pmatrix} (1, p)$. Since the two quadratic forms $Ax \cdot x$, $Px \cdot x$ are equal on the hyperplane $\{x_0 + p \cdot x_1 = 0\}$, there exists a constant $k \geq 0$ such that

$$Ax \cdot x = kPx \cdot x = k|x_0 + p \cdot x_1|^2$$

for each $x = (x_0, x_1) \in \mathbf{R}^{n+1}$. Hence and from the explicit form of the Γ-limit (see (16), for $u(t,x) = t$) we get $k = Ae_1 \cdot e_1 \leq J(u) \leq 1$. ■

REMARK 4.3: We can repeat the proof of Theorem 3.3 and we obtain that the element a_{00} of the matrix A is positive iff T possesses an absolutely continuous measure with L^2_{loc} density.

Now, we introduce the notion of the limit directions.

Definition 4.1: *We say that* $\alpha \in \boldsymbol{R}^n$ *is a limit direction for the equation* $x' = f(t, x)$, $f \in C^1_{per}(\boldsymbol{R}^{n+1})^n$ *iff there exist sequences* $(\varepsilon_k) \subset (\varepsilon)$, $(y_k) \subset \boldsymbol{R}^n$ *and* $y_0 \in \boldsymbol{R}^n$ *such that*

$$T_t^{(\varepsilon_k)}(y_k) \to \alpha t + y_0$$

uniformly with respect to t *on compact sets. The set of all limit directions is denoted by* $\mathcal{D}$.

Remark 4.4: $\rho_1(T) \subset \mathcal{D}$ ($\rho_1(T)$ is defined in (7)). This fact follows immediately from the proof of Proposition 4.3.

We have the following

Proposition 4.4: *Let* $f \in C^1_{per}(\boldsymbol{R}^{n+1})^n$. *Then* $\mathcal{D} \subset \rho(T)$.

Proof: If $p \in \mathcal{D}$, then there exist $(\varepsilon_k) \subset (\varepsilon)$, $(y_k) \subset \boldsymbol{R}^n$ and $y_0 \in \boldsymbol{R}^n$ such that

$$\varepsilon_k T_{1/\varepsilon_k}\left(\left(\frac{y_k}{\varepsilon_k}\right)^{-}\right) = \varepsilon_k T_{1/\varepsilon_k}\left(\frac{y_k}{\varepsilon_k}\right) - \varepsilon_k\left[\frac{y_k}{\varepsilon_k}\right] = T_1^{(\varepsilon_k)}(y_k) - \varepsilon_k\left[\frac{y_k}{\varepsilon_k}\right] \to p\,.$$

Hence and from the relation

$$\frac{T_{\tau_n}(p_n)}{\tau_n} - \frac{T_{[\tau_n]}(p_n)}{[\tau_n]} \to 0\,,$$

if $T_{[\tau_n]}(p_n)/\tau_n$ is converging, we have that p belongs to $\rho(T)$. ∎

We conclude this section with the following important

Example 4.2: Let $n = 2$, $g \in C^1_{per}(\boldsymbol{R})$, $g \not\equiv$ const. We consider the following system of ordinary differential equations

$$\text{(31)} \qquad \begin{cases} x' = 0\,, \\ y' = g\left(\dfrac{x}{\varepsilon}\right), \qquad x(0) = x_0\,,\ y(0) = y_0\,,\ \varepsilon > 0\,. \end{cases}$$

The solutions to (31) are given by the formula

$$T_t^{(\varepsilon)}(x_0, y_0) = \left(x_0, y_0 + tg\left(\frac{x_0}{\varepsilon}\right)\right).$$

Hence, we can observe that $\mathcal{D} = \rho(T) = \{(0, r) \in \boldsymbol{R}^2 \mid r \in \operatorname{Im} g\}$. Then the equations (31) do not G-converge.

Moreover, we can calculate explicitly the set $\{\psi = 0\}$ and we will see that it is equal to $\rho(T)$. Namely, we suppose $\xi \in \boldsymbol{R}^2$, $\xi = (\xi_1, \xi_2)$, $\psi(\xi) = 0$. From formula (13), we have $\psi_T(\xi) \geqslant \xi_1^2$ for each $T > 0$. Hence $\xi_1 = 0$. Now, we have

$$\psi_T(0, \xi_2) \geqslant \inf\left\{\left|\xi_2 - \frac{1}{T}\int_0^T g(x(s))\,ds\right|^2 : x(0) = 0,\ x(T) = \xi T,\ x \in C^1([0, T])^2\right\}.$$

Since $\lim\limits_{T\to+\infty} \psi_T(\xi) = 0$, we obtain $\xi_T \in \operatorname{Im} g$. This proves $\{\psi = 0\} \subset \rho(T)$ and by Proposition 4.3, we have that the two sets are identical.

The $\Gamma(s - L^2(\Omega)^-)$-limit of the sequence

$$G_\varepsilon(u) = \int_\Omega \left(u_t + g\left(\frac{x}{\varepsilon}\right)\cdot u_y\right)^2 dt\,dx\,dy$$

has the form

$$G_0(u) = \int_\Omega ((u_t + \bar{g}u_y)^2 + (\gamma - \bar{g}^2)u_y^2)\,dt\,dx\,dy\,, \tag{32}$$

where $\bar{g} = M(g)$, $\gamma = M(g^2)$.

The $\Gamma(w - L^2(\Omega)^-)$-limit of the sequence $(G_\varepsilon)_\varepsilon$, studied in [1], has no integral representation and hence it is different from the limit (32).

The solutions of the corresponding first order partial differential equation

$$\begin{cases} u_t + g\left(\dfrac{x}{\varepsilon}\right)u_y = 0\,, \\ u(0,x,y) = \phi(x,y)\,, \qquad \phi \in C_0^1(\boldsymbol{R}^2) \end{cases}$$

are of the form $u^\varepsilon(t,x,y) = \phi(x, y - tg(x/\varepsilon))$. The sequence $(u_\varepsilon)_\varepsilon$ converges weakly in $L^2_{loc}(\boldsymbol{R}^3)$ to the function

$$u(t,x,y) = \int_I \phi(x, y - tg(s))\,ds\,,$$

even if the corresponding sequence of the equations of characteristics does not G-converge.

5. - Open questions

In this section we indicate the main problems related to our results but still not solved in the n-dimensional case. For other conjectures see [9].

Problem 1: Is it true that for a given $f \in C^1_{per}(\boldsymbol{R}^{n+1})^n$, $\phi \in C^1_0(\boldsymbol{R}^n)$ the solutions of (30) converge weakly in $L^2_{loc}(\boldsymbol{R}^{n+1})$ and moreover, that this weak limit has a representation of the type

$$\int_{\boldsymbol{R}^n} \phi(x - t\xi)\,d\mu(\xi)\,,$$

where μ is a probability measure which depends only on the function f?

Problem 2: Is the set $\{\psi = 0\}$ equal to the set $\operatorname{conv} \rho(T)$, where ψ is the convex function given by the formula (12)?

PROBLEM 3: Does the G-convergence imply the strong G-convergence for all functions $f \in C^1_{per}(R^{n+1})^n$?

PROBLEM 4: Which are the possible matrices A given by (16), corresponding to the functions in $C^1_{per}(R^{n+1})^n$? In particular, we want to find examples where $\det(A) \neq 0$.

Some answers to these problems are given by Peirone in [25].

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